Appendix A

A.1

(a) $[A] + [B] = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -3 & 12 \end{bmatrix}$

(b) $[A] + [C]$, Nonsense, $[A]$ and $[C]$ not same order

(c) $[A]\,[C]^T$, Nonsense, columns $[A] \neq$ rows $[C]^T$

(d) $[D]\,[E] = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$

$$= \begin{Bmatrix} 3(1) + 1(2) + 2(3) \\ 1(1) + 4(2) + 0(2) \\ 2(1) + 0(2) + 3(3) \end{Bmatrix} = \begin{Bmatrix} 11 \\ 9 \\ 11 \end{Bmatrix}$$

(e) $[D]\,[C]$, Nonsense, columns $[D] \neq$ rows $[C]$

(f) $[C]\,[D] = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 3(3) + (1)(1) + 0 & 3 + 4 + 0 & 6 + 0 + 0 \\ -3 + 0 + 6 & -1 + 0 + 0 & -2 + 0 + 9 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 7 & 6 \\ 3 & -1 & 7 \end{bmatrix}$$

A.2 $[A] = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} \quad [A]^{-1} = \dfrac{[C]^T}{|[A]|}$

$$C_{11} = (-1)^{1+1}(4) = 4, \qquad C_{12} = (-1)^{1+2}(-1) = 1$$

$$C_{21} = (-1)^{2+1}(0) = 0, \qquad C_{22} = (-1)^{2+2}(1) = 1$$

$$[C] = \begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix}$$

$$|[A]| = A_{11}C_{11} + A_{12}C_{12}$$

$$= (1)(4) + (0)(1) = 4$$

$$[C]^T = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \; [A]^{-1} = \frac{\begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}}{4} = \begin{bmatrix} 1 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Verify by multiplying $[A]\,[A]^{-1} = [I]$

A.3 $[D]^{-1} = \dfrac{[C]^T}{|[D]|}$

$$[D] = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 12 & -3 & -8 \\ -3 & 5 & 2 \\ -8 & 2 & 11 \end{bmatrix}$$

$$|\,[D]\,| = 12(3) + (-3)(1) + (-8)(2) = 17$$

$$[D]^{-1} = \frac{1}{17}\begin{bmatrix} 12 & -3 & -8 \\ -3 & 5 & 2 \\ -8 & 2 & 11 \end{bmatrix}$$

A.4 Nonsense

A.5 $[B] = \begin{bmatrix} 2 & 0 \\ -2 & 8 \end{bmatrix}$

(1) $\left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ -2 & 8 & 0 & 1 \end{array}\right]$ divide 1st row by 2

(2) $\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 0 \\ -2 & 8 & 0 & 1 \end{array}\right]$ multiply 1st row by 2 and add to row 2

(3) $\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 8 & 1 & 1 \end{array}\right]$ divide 2nd row by 8

(4) $\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{8} & \frac{1}{8} \end{array}\right]$ $\therefore [B]^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{8} \end{bmatrix}$

A.6 $[D]^{-1}$ by row reduction

$\left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$ divide row 1 by 3

$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$ subtract row 1 from 2

$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 3\frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$. multiply row 1 by 2 and subtract from row 3

$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 3\frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} & 1 & 0 \\ 0 & \frac{-2}{3} & \frac{5}{3} & \frac{-2}{3} & 0 & 1 \end{array}\right]$. multiple row 2 by $\frac{2}{11}$ and add to row 3

$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 3\frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} & 1 & 0 \\ 0 & 0 & \frac{51}{33} & \frac{-24}{33} & \frac{2}{11} & 1 \end{array}\right]$. multiply row 2 by $\frac{3}{11}$ and row 3 by $\frac{33}{51}$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{-2}{11} & \frac{-1}{11} & \frac{3}{11} & 0 \\ 0 & 0 & 1 & \frac{-24}{51} & \frac{6}{51} & \frac{33}{51} \end{array}\right]. \text{ multiply row 3 by } \tfrac{2}{11} \text{ and add to row 2}$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & \frac{-3}{17} & \frac{5}{17} & \frac{2}{17} \\ 0 & 0 & 1 & \frac{-24}{51} & \frac{6}{51} & \frac{33}{51} \end{array}\right]. \text{ multiply row 3 by } \tfrac{2}{3} \text{ and subtract from row 1}$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & 0 & \frac{11}{17} & \frac{-4}{51} & \frac{-22}{51} \\ 0 & 1 & 0 & \frac{-3}{17} & \frac{5}{17} & \frac{2}{17} \\ 0 & 0 & 1 & \frac{-24}{51} & \frac{6}{51} & \frac{33}{51} \end{array}\right]. \text{ multiply row 2 by } \tfrac{1}{3} \text{ and subtract from row 1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{12}{17} & \frac{-3}{17} & \frac{-8}{17} \\ 0 & 1 & 0 & \frac{-3}{17} & \frac{5}{17} & \frac{2}{17} \\ 0 & 0 & 1 & \frac{-8}{17} & \frac{2}{17} & \frac{11}{17} \end{array}\right]$$

$$\therefore\ [D]^{-1} = \frac{1}{17}\begin{bmatrix} 12 & -3 & -8 \\ -3 & 5 & 2 \\ -8 & 2 & 11 \end{bmatrix}$$

A.7 Show that $([A]\ [B])^T = [B]^T\ [A]^T$ by using

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad [B] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$([A]\ [B]) = \begin{bmatrix} a_{11}(b_{11}) + a_{12}(b_{21}) & a_{11}(b_{12}) + a_{12}(b_{22}) & a_{11}(b_{13}) + a_{12}(b_{23}) \\ a_{21}(b_{11}) + a_{22}(b_{21}) & a_{21}(b_{12}) + a_{22}(b_{22}) & a_{21}(b_{13}) + a_{22}(b_{23}) \end{bmatrix}$$

$$([A]\ [B])^T = \begin{bmatrix} a_{11}(b_{11}) + a_{12}(b_{21}) & a_{21}(b_{11}) + a_{22}(b_{21}) \\ a_{11}(b_{12}) + a_{12}(b_{22}) & a_{21}(b_{12}) + a_{22}(b_{22}) \\ a_{11}(b_{13}) + a_{12}(b_{23}) & a_{21}(b_{13}) + a_{22}(b_{23}) \end{bmatrix}$$

$$[B]^T = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{23} \end{bmatrix} \qquad [A]^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$[B]^T\ [A]^T = \begin{bmatrix} b_{11}(a_{11}) + b_{21}(a_{12}) & b_{11}(a_{21}) + b_{21}(a_{22}) \\ b_{12}(a_{11}) + b_{22}(a_{12}) & b_{12}(a_{21}) + b_{22}(a_{22}) \\ b_{13}(a_{11}) + b_{23}(a_{12}) & b_{13}(a_{21}) + b_{23}(a_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(b_{11}) + a_{12}(b_{21}) & a_{21}(b_{11}) + a_{22}(b_{21}) \\ a_{11}(b_{12}) + a_{12}(b_{22}) & a_{21}(b_{12}) + a_{22}(b_{22}) \\ a_{11}(b_{13}) + a_{12}(b_{23}) & a_{21}(b_{13}) + a_{22}(b_{23}) \end{bmatrix}$$

Answer : $([A]\ [B])^T = [B]^T\ [A]^T$

A.8 $[T] = \begin{bmatrix} C & S \\ -S & C \end{bmatrix}$

$$[C] = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \qquad [C]^T = \begin{bmatrix} C & -S \\ S & C \end{bmatrix}$$

$$|\,[T]\,| = C^2 + S^2 = 1$$

$$[T]^{-1} = \frac{[C]^T}{|[T]|} = \begin{bmatrix} C & -S \\ S & C \end{bmatrix}$$

and

$$[T]^T = \begin{bmatrix} C & -S \\ S & C \end{bmatrix}$$

$\therefore \quad [T]^T = [T]^{-1}$ and T is an orthogonal matrix

A.9 Show $\{X\}^T [A] \{X\}$ is symmetric. Given

$$\{X\} = \begin{bmatrix} x & y \\ 1 & x \end{bmatrix}, [A] = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$\{X\}^T = \begin{bmatrix} x & 1 \\ y & x \end{bmatrix}$$

$$\{X\}^T [A] \{X\} = \begin{bmatrix} x & 1 \\ y & x \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x & y \\ 1 & x \end{bmatrix}$$

$$= \begin{bmatrix} ax+b & bx+c \\ ay+bx & by+cx \end{bmatrix}\begin{bmatrix} x & y \\ 1 & x \end{bmatrix}$$

$$= \begin{bmatrix} ax^2+bx+bx+c & axy+by+bx^2+cx \\ axy+bx^2+by+cx & ay^2+bxy+bxy+cx^2 \end{bmatrix}$$

as the 1–2 term = 2–1 term $\{X\}^T [A] \{X\}$ is symmetric.

A.10 Evaluate $[K] = \int_0^L [B]^T \; E \, [B] \, dx$, $[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$

$$[K] = \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} E \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} dx$$

$$[K] = \int_0^L \begin{bmatrix} \frac{1}{L^2} & \frac{-1}{L^2} \\ \frac{-1}{L^2} & \frac{1}{L^2} \end{bmatrix} E \, dx$$

$$[K] = E \begin{bmatrix} \frac{1}{L} & \frac{1}{L} \\ \frac{-1}{L} & \frac{1}{L} \end{bmatrix} = \frac{E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(Should multiply by A to get actual $[K]$ for a bar)

A.11 The following integral represents the strain energy in a bar

$$U = \frac{A}{2}\int_0^L \{d\}^T \; [B]^T [D] \; [B] \; \{d\} \; dx$$

where $\quad \{d\} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} \qquad [B] = \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} \qquad [D] = E$

Show that $\frac{dU}{d\{d\}}$ yields $[k]\,\{d\}$, where $[k]$ is the bar stiffness matrix given by

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\{d\}^T = [d_1\ d_2]$ $\qquad$ $[B]^T = \begin{Bmatrix} \frac{-1}{L} \\ \frac{1}{L} \end{Bmatrix}$

$$U = \frac{A}{2}\int_0^L \{d\}^T\ [B]^T\ [D]\ [B]\ \{d\}\ dx$$

$$U = \frac{AL}{2}\ \{d\}^T\ [B]^T\ [D]^T\ [B]\ \{d\} = \frac{AL}{2}\ [d_1 \quad d_2]\ \begin{Bmatrix} \frac{-1}{L} \\ \frac{1}{L} \end{Bmatrix} [E] \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$U = \frac{AL}{2}\left[\frac{d_2 - d_1}{L}\right][E]\begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$U = \frac{AEL}{2}\left[\frac{d_2 - d_1}{L}\right]\begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \frac{AEL}{2}\begin{bmatrix} \frac{d_1 - d_2}{L^2} & \frac{-d_1 + d_2}{L^2} \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$U = \frac{AEL}{2}\left[\frac{d_1^2 - d_1 d_2 - d_1 d_2 + d_2^2}{L^2}\right] = \frac{AE}{2L}\ [\, d_1^2 - 2d_1 d_2 + d_2^2\,]$$

$$\frac{dU}{d\{d\}} = \begin{Bmatrix} \frac{\partial U}{\partial d_1} \\ \frac{\partial U}{\partial d_2} \end{Bmatrix} = \begin{Bmatrix} \frac{AE}{2L}(2d_1 - 2d_2) \\ \frac{AE}{2L}(2d_2 - 2d_1) \end{Bmatrix} = \frac{AE}{L}\begin{Bmatrix} d_1 - d_2 \\ -d_2 - d_1 \end{Bmatrix} = \frac{AE}{L}\begin{Bmatrix} d_1 - d_2 \\ -d_1 + d_2 \end{Bmatrix}$$

$$= \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$\frac{dU}{d\{d\}} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = [k]\ \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = [k]\ \{d\} \text{ knowing that } [k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Thus $\dfrac{dU}{d\{d\}} = [k]\ \{d\}$